GEOMETRY
COMPUTING TRANSFORMATIONS WITH VECTORS

PAUL L. BAILEY

1. Vectors

A *vector* indicates how to move a certain distance in a certain direction. In coordinates, this can by specified by indicating how far to the right to go, and how far up to go.

We write $\vec{v} = \langle a, b\rangle$ to specify that $\vec{v}$ is the vector that goes right by a and up by b. Of course, going right by $-a$ means going left by a, and going up by $-b$ means going down by b.

If A and B are points, and $\vec{v}$ is the vector that indicates how to move from A to B, we write $\vec{v} = \overrightarrow{AB}$, and $B = A + \vec{v}$. Thus we may add a vector to a point to get another point.

We can compute $\overrightarrow{AB}$; it is the difference between the points. That is, the vector from A to B is $\overrightarrow{AB} = A - B$. We give an example.

You can subtract points to get a vector:

$$(5, 2) - (1, 4) = \langle 4, -2\rangle.$$

This means that to get from point $(1, 4)$ to point $(5, 2)$, we must go right by 4 and down by 2. This may be written

$$(1, 4) + \langle 4, -2\rangle = (5, 2).$$

Thus, we may add a vector to a point.

If we take the point $(-3, 17)$ and go right by 4 and down by 2, we get to the point $(1, 15)$; this is written $(-3, 17) + \langle 4, -2\rangle = (1, 15)$. In this case, we have $(1, 15) - (-3, 17) = \langle 4, -2\rangle$.

The difference between $(5, 2)$ and $(1, 4)$ is the same as the difference between $(1, 15)$ and $(-3, 17)$; thus the vector from $(1, 4)$ to $(5, 2)$ is *the same vector* as the vector from $(-3, 17)$ to $(1, 15)$.

Finally, we may add vectors:

$$\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle.$$

This means that if we go right by a and up by b, and then go right by c and up by d, that is the same as going right by $a + c$ and then up by $b + d$.

The *zero vector* is $\vec{0} = \langle 0, 0\rangle$; this has the property that if added to any other vector, the result leaves the other vector unchanged:

$$\vec{v} + \vec{0} = \vec{v}.$$

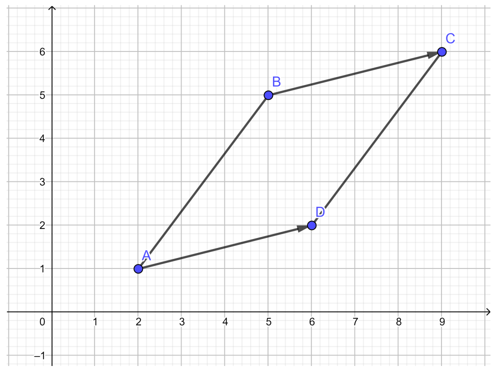
The *negative vector* of $\vec{v} = \langle a, b\rangle$ is $-\vec{v} = \langle -a, -b\rangle$. This has the property that $\vec{v} + (-\vec{v}) = \vec{0}$.

Date: May 10, 2021.

2. Finding Points

Problem 1. Let $ABCD$ be a parallelogram, with $A = (2,1)$, $B = (5,5)$, and $C = (9,6)$. Find D.

Solution. First, draw this.



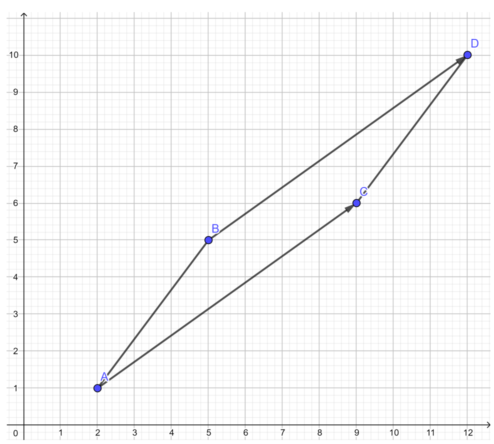
To get from B to C, we go right 4 and up 1. That is,

$$\overrightarrow{BC} = C - B = (9,6) - (5,5) = \langle 4,1 \rangle.$$

So, the parallel arrow of the same length, starting from A, ends at

$$D = A + \overrightarrow{BC} = (2,1) + \langle 4,1 \rangle = (6,2).$$

There is another solution. Instead of connection $\overrightarrow{BC}$, use A and C.



That is, compute

$$\overrightarrow{AC} = C - A = (9,6) - (2,1) = \langle 7,5 \rangle.$$

Then let

$$D = B + \overrightarrow{AC} = (5,5) + \langle 7,5 \rangle = (12,10).$$

□

3. Translations

A *translation* is a transformation obtained by adding a single vector to every point in the plane. Thus, if T is translation by the vector $\vec{v} = \langle a, b\rangle$, then the formula for T is

$$T(x, y) = (x, y) + \langle a, b\rangle = (x + a, y + b).$$

The vector of a composition of translations is the sum of the translation vectors.

Example 1. Find a formula for translation by $\langle 3, -7\rangle$.

Solution. Let $\vec{v} = \langle 3, -7\rangle$ and let T be translation by $\vec{v}$. Then

$$T(x, y) = (x, y) + \langle 3, -7\rangle = (x + 3, y - 7).$$

□

Example 2. Let $R(x, y) = (x - 3, y + 2)$ and $S(x, y) = (x + 11, y + 10)$. Find the vector of $S \circ R$.

Solution. We have

$$S\circ R(x, y) = S(R(x, y)) = S(x-3, y+2) = ((x-3)+11, (y+2)+10) = (x+8, y+12).$$

Thus the vector of translation is $\langle 8, 12\rangle$.

From another perspective, let $\vec{v} = \langle -3, 2\rangle$ and $\vec{w} = \langle 11, 10\rangle$. Then $\vec{v}$ is the vector of R, and $\vec{w}$ is the vector of S. Therefore, the vector of $S \circ R$ is

$$\vec{v} + \vec{w} = \langle -3, 2\rangle + \langle 11, 10\rangle = \langle 8, 12\rangle.$$

□

4. Rotations

4.1. **Rotation by 90°.** Rotation by and angle θ by *counterclockwise* by θ; this is convention.

Draw an arrow starting at the origin; the tip of the arrow is at $\langle x, y\rangle$. Draw a perpendicular arrow of the same length, counterclockwise; the resulting tip will be at $\langle -y, x\rangle$.

If $\vec{v} = \langle x, y\rangle$, the *perpendicular vector* is $\vec{v}^{\perp} = \langle -y, x\rangle$.

Let $R(x, y)$ denote rotation of the point $A = (x, y)$ around the point $K = (h, k)$. Compute $R(x, y)$ as follows:

(1) Form the vector $\overrightarrow{KA} = A - K = \langle x - h, y - k\rangle$.
(2) Form the vector $\overrightarrow{KA}^{\perp} = \langle k - y, x - h\rangle$.
(3) Add this to K to get $R(x, y) = K + \overrightarrow{KA}^{\perp} = (h + k - y, k + x - h)$.

Thus

$$R(x, y) = \langle h + k - y, x + y - h\rangle.$$

Example 3. Let R be rotation by 90° about $K = (2, 5)$. Let $A = (11, 3)$. Find $R(A)$.

Solution. Follow that algorithm.

(1) $\overrightarrow{KA} = A - K = (11, 3) - (2, 5) = \langle 9, -2\rangle$.
(2) $\overrightarrow{KA}^{\perp} = \langle 2, 9\rangle$.
(3) $R(A) = K + \overrightarrow{KA}^{\perp} = (2, 5) + \langle 2, 9\rangle = (4, 14)$.

□

Example 4. Let R be rotation by $90°$ about $K = (2, 5)$. Find $R(x, y)$.

Solution. Follow that algorithm, now with a variable point. Let $A = (x, y)$ be the variable point.

(1) $\overrightarrow{KA} = A - K = (x, y) - (2, 5) = \langle x - 2, y - 5\rangle$.
(2) $\overrightarrow{KA}^{\perp} = \langle 5 - y, x - 2\rangle$.
(3) $R(A) = K + \overrightarrow{KA}^{\perp} = (2, 5) + \langle 5 - y, x - 2\rangle = (7 - y, x + 3)$.

□

Example 5. Let R be rotation by $90°$ about $K = (2, 5)$, and let S be rotation by $90°$ about $L = (4, 1)$. Compute a formula for the composition $S \circ R$.

Solution. First, we know that $R(x, y) = (7 - y, x + 3)$.

Now, compute a formula for S. Let $A = (x, y)$.

(1) $\overrightarrow{LA} = A - L = (x, y) - (4, 1) = \langle x - 4, y - 1\rangle$.
(2) $\overrightarrow{LA}^{\perp} = \langle 1 - y, x - 4\rangle$.
(3) $S(A) = L + \overrightarrow{LA}^{\perp} = (4, 1) + \langle 1 - y, x - 4\rangle = (5 - y, x - 3)$.

Next, compute $S \circ R$:

$$S \circ R(A) = S(R(x, y)) = S(7 - y, x + 3) = (5 - (x + 3), (7 - y) - 3) = (2 - x, 4 - y).$$

□

Example 6. Let R be rotation by $90°$ about $K = (2, 5)$, and let S be rotation by $90°$ about $L = (4, 1)$. Then $S \circ R$ is a rotation by $180°$. Find the center of $S \circ R$.

Solution. Let $T = S \circ R$. From Example 6, $T(x, y) = (2 - x, 4 - y)$.

The center of the rotation is the unique fixed point. We know that (x, y) is a fixed point of T if and only if $T(x, y) = (x, y)$. So, we solve this equation:

$$(x, y) = (2 - x, 4 - y) \text{ if and only if } x = 2 - x \text{ and } y = 4 - y.$$

From this, we see that $2x = 2$ so $x = 1$, and $2y = 4$ so $y = 2$. Thus the center is $(1, 2)$. □

4.2. **Rotation by $180°$.** Rotation by $180°$ around a point $K = (h, k)$ can be thought of as "reflection through the point K"; that is, the image of $A = (x, y)$ is the point on the line through A and K which is on the other side of K at distance AK. To compute this, let R denote rotation about K by $180°$.

(1) Compute the vector $\overrightarrow{KA} = A - K = \langle x - h, y - k$.
(2) Subtract this from K to get $R(A) = K - \overrightarrow{KA} = (h, k) - \langle x - h, y - k\rangle = (2h - x, 2k - y)$.

Example 7. Let R be rotation about $(3, 7)$ by $180°$. Find a formula for R.

Solution. Let's not use the formula, but instead, practice the idea. Follow the algorithm.

Let $K = (3, 7)$ and let $A = (x, y)$.

(1) $\overrightarrow{KA} = A - K = \langle x - 3, y - 7$.
(2) $R(x, y) = K - \overrightarrow{KA} = (3, 7) - \langle x - 3, y - 7\rangle = (3 - (x - 3), 7 - (y - 7)) = (6 - x, 14 - y)$.

□

Example 8. Let R be rotation about $(3,7)$ by $180°$, and let S be rotation about $(9,-4)$ by $180°$. Find a formula for $R \circ S$, and characterize it as a transformation.

Solution. First, we know that $R(x,y) = (6-x, 14-y)$.

Next, our formula says that the image of (x,y) after rotation around (h,k) by $180°$ is $(2h-x, 2k-y)$. Thus, $S(x,y) = (18-x, -8-y)$.

Compute

$$R \circ S(x,y) = R(S(x,y)) = R(18-x, -8-y) = (6-(18-x), 14-(-8-y)) = (x-12, y-6).$$

This is translation by the vector $\langle -12, -6 \rangle$. □

Example 9. Let R be rotation by $90°$ about $K = (5,2)$, and let S be rotation about $L = (3,4)$ by $180°$.

(a) What is $R(S(6,2))$?
(b) $R \circ S$ is rotation by $270°$ about what point?

Solution. Let $A = (x,y)$.

Find the formula for R:

(1) $\overrightarrow{KA} = A - K = (x,y) - (5,2) = \langle x-5, y-2 \rangle$.
(2) $\overrightarrow{KA}^{\perp} = \langle 2-y, x-5 \rangle$.
(3) $R(x,y) = K + \overrightarrow{KA}^{\perp} = (5,2) + \langle 2-y, x-5 \rangle = (7-y, x-3)$.

Find a formula for S:

(1) $\overrightarrow{LA} = A - L = \langle x-3, y-4$.
(2) $S(x,y) = L - \overrightarrow{LA} = (3,4) - \langle x-3, y-4 \rangle = (3-(x-3), 4-(y-4)) = (6-x, 8-y)$.

Compute the composition:

$$R \circ S(A) = R(S(x,y)) = R(6-x, 8-y) = (7-(8-y), (6-x)-3) = (y-1, 3-x).$$

(a) Compute $R(S(6,2)) = (2-1, 3-6) = (1,-3)$.
(b) Find the fixed point by solving $(x,y) = (y-1, 3-x)$. We have $x = y-1$ and $y = 3-x$. Substituting the second equation into the first gives $x = (3-x)-1 = 2-x$, so $2x = 2$, whence $x = 1$. Thus $y = 3-1 = 2$. The center is $(1,2)$.

□

5. Reflections

Let T be reflection across the line $y = mx + b$. Let A be any point. We compute $T(A)$ as follows.

(1) Find the equation of the line through A and perpendicular to $y = mx + b$.
(2) Find the point M which is the intersection of these two lines.
(3) Compute the vector $\overrightarrow{MA}$.
(4) Compute the point $T(A) = M - \vec{MA}$.

Example 10. Find the reflection of $A = (9, 4)$ across the line $3x + 2$.

Solution. We follow our steps.

(1) Find the equation of the line through A and perpendicular to $y = mx + b$.
The slope of the perpendicular line is $-\dfrac{1}{3}$, so this line is

$$y = -\frac{1}{3}(x - 9) + 4 = -\frac{1}{3}x + 7.$$

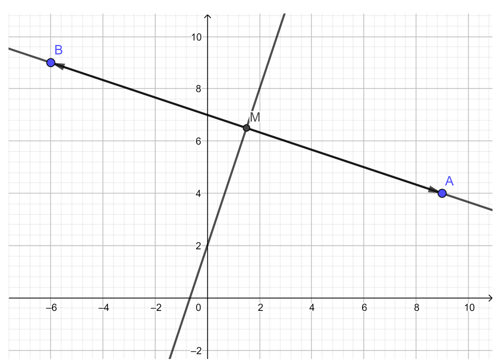
(2) Find the point M which is the intersection of these two lines.
Solve $3x + 2 = -\dfrac{1}{3}x + 7$. Subtract 2 to get $3x = -\dfrac{1}{3}x + 5$. Multiply by 3 to get $9x = -x + 15$. Then $10x = 15$, so $x = \dfrac{15}{10} = \dfrac{3}{2}$. Thus $y = 3 \cdot \dfrac{3}{2} + 2 = \dfrac{13}{2}$.
So, $M = \left(\dfrac{3}{2}, \dfrac{13}{2}\right)$.

(3) Compute the vector $\overrightarrow{MA}$.
We have

$$\overrightarrow{MA} = A - M = (9, 4) - (\frac{3}{2}, \frac{13}{2}) = \left\langle \frac{15}{2}, -\frac{5}{2} \right\rangle.$$

(4) Compute the point $T(A) = M - \vec{MA}$.
Thus

$$T(9, 4) = M - \vec{AM} = \left(\frac{3}{2}, \frac{13}{2}\right) - \left\langle \frac{15}{2}, -\frac{5}{2} \right\rangle = \left(-\frac{12}{2}, \frac{18}{2}\right) = (-6, 9).$$



□

Department of Mathematics, Veritas Preparatory Academy
Email address: pbailey@veritasprepacademy.org